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ONE-DIMENSIONAL HYDROELASTIC OSCILLATIONS OF THREE-LAYERED PLATES*

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A method is proposed for calculating one-dimensional (using modes of cylindrical bending) and axially-symmetric degenerate vibrations of three-layered plates which are clamped in an infinitely rigid screen on the boundary of separation of fluid media. The account is presented as it applies to axially-symmetric vibrations, but the changes which have to be introduced in solving the planar problem are indicated in the appropriate places. The dependence of the resonant frequencies and of the amplitudes of the vibrations on the geometrical and physical parameters of the layers, the asymmetry in the structure of the plate and the damping properties of the filler are investigated. The special features of the hydroelastic vibrations of three-layered plates compared with the vibrations of homogeneous plates are noted. The latter have been treated in a number of papers /1, 2/**

Let us consider the degenerate axially-symmetric vibrations of a circular three-layered plate which has been clamped in an infinitely rigid screen on the boundary of separation between fluid half spaces. Omitting the time factor $e^{-i\omega t}$, we shall describe the bending of the plate by means of equations constructed by invoking the extensively used /3/ hypotheses concerning the incompressibility of the filler in the transverse direction and the linear distribution of the tangential displacements throughout its thickness

$$\begin{gathered}
b_1\frac{d}{dr}\nabla^2 w - b_2\omega^2\frac{dw}{dr} + b_3\frac{dw}{dr} + b_4D^2\varphi - b_5\omega^2\varphi - \\
b_3\varphi = ht,\ c_1\nabla^2\nabla^2 w - c_2\omega^2\nabla^2 w - b_3\nabla^2 w - c_3\omega^2 w + b_3 D\varphi + \\
b_1\nabla^2 D\varphi - b_2\omega^2 D\varphi = -(q+p) + \delta Dt \\
D = \frac{d}{dr} + \frac{1}{r},\quad D^2 = \frac{d}{dr}D,\quad \nabla^2 = D\frac{d}{dr},\quad q = q_1 - q_2, \\
p = p_1 - p_2,\quad t = t_1 + t_2 \\
b_1 = -E_1h\delta^2,\ b_2 = \rho_1 h\delta^2,\ b_3 = -2Gh,\ b_4 = -{}^2/{}_3Eh^3 - 2E_1h^2\delta \\
b_5 = {}^2/{}_3\rho h^3 + 2\rho_1h^2\delta,\ c_1 = -{}^2/{}_3E_1\delta^3,\ c_2 = {}^2/{}_3\rho_1\delta^3,\ c_3 = -2(\rho h + \rho_1\delta) \\
E_1 = E_1'/(1-\nu_1^2),\ E = E'/(1-\nu^2)
\end{gathered} \tag{1}$$

Here w is the deflection of the plate, φ is a function describing the rotation of sections of the filler, q_1, q_2, t_1, and t_2 are the normal and tangential loads acting on the upper and lower covers, p is the difference between the acoustic pressures on the two sides of the plate, E_1', ν_1, and ρ_1 are the modulus of elasticity, Poisson's ratio and the density of the cover material of thickness δ and E', ν, ρ and G are the same characteristics and the shear modulus of the filler material with a thickness of $2h$.

We note that, when the differential operators in (1) are replaced by the corresponding derivatives with respect to the x-coordinate and, also, when the tangential loads t_1 and t_2 are equal to zero, we obtain the equations for the bending of a three-layered plate-strip /4/.

We shall seek a solution of Eqs.(1) which satisfies the condition that the deflection is equal to zero on the plate contour in the form of a series in Bessel functions

$$w = \Sigma_k w_k J_0(x_k r/a),\ \varphi = \Sigma_k \varphi_k J_1(x_k r/a) \tag{2}$$

where x_k are the roots of the equation $J_0(x) = 0$, a is the radius of the plate and Σ_k denotes summation over k from 1 to ∞.

In the case of a planar problem, expressions (2) are replaced by

$$w = \Sigma_k w_k \sin\alpha_k x,\ \varphi = \Sigma_k\varphi_k\cos\alpha_k x,\ \alpha_k = k\pi/l$$

where l is the width of the plate-strip.

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**Golovanov V.A., Muzychenko V.V., Peker F.N. and Popov A.L., Scattering and radiation of sound by elastic shells in a fluid, Preprint 261, Inst. Problem Mekhaniki Akad. Nauk SSSR, Moscow, 1985 and the bibliography given there.

In order to satisfy the remaining boundary conditions, we apply an additional bending moment M and a tangential load T to the contour. The overall load on the plate will therefore consist of the exciting load $q(r)$, the acoustic pressure $p(r)$, loads with an intensity q', applied over the area of a ring of width ε_1 at a distance ε from the edge of the plate and the tangential load t which acts over a ring of width ε on the edge of the plate.

In order to pass to the edge bending moment and the edge load, we will assume that

$$q'\varepsilon_1 \to p',\ p'\varepsilon \to M,\ \varepsilon t \to T \text{ when } \varepsilon_1,\ \varepsilon \to 0 \tag{3}$$

We expand the above-mentioned components of the load on the plate in a series of Bessel functions. Here, we approximate the unknown function $p(r)$ in the interval $[0, a]$ by means of a piecewise-linear function. By making use of the orthogonality of Bessel functions, and taking account of formulae (3) and the approximation which has been adopted for $p(r)$, we find the expansion coefficients for the above-mentioned loads.

$$q_k = \frac{2}{a^2 J_1^2(x_k)} \int_0^a q(r) J_0\left(x_k \frac{r}{a}\right) r\,dr \tag{4}$$

$$q_k' = -\frac{2Mx_k}{a^2 J_1(x_k)}, \quad t_k = \frac{2T}{aJ_1(x_k)}, \quad p_k = \Sigma_j p_j z_{kj}$$

$$z_{kj} = \frac{2}{a^2 J_1^2(x_k)} \left[\int_{(j-1)\Delta r}^{j\Delta r} \frac{j\Delta r - r}{\Delta r} J_0\left(x_k \frac{r}{a}\right) r\,dr - \right.$$

$$\left. \int_{(j-2)\Delta r}^{(j-1)\Delta r} \frac{(j-2)\Delta r - r}{\Delta r} J_0\left(x_k \frac{r}{a}\right) r\,dr \right]; \quad j = 2, 3, \ldots N; \quad k = 1, 2, \ldots$$

Here N is the number of segments into which the plate is subdivided, p_j are the acoustic pressures on the boundaries of these segments and Σ_j denote summation over j from 1 to $N+1$. When $j = 1$ and $j = N+1$, expressions for z_{kj} are obtained from the general formula after the second and first integral, respectively, have been discarded. In solving the planar problem the components of the load are represented by trigonometric series.

By substituting expansions (2) and (4) into Eqs.(1) we can find the coefficients

$$w_k = \frac{2x_k}{a^2 J_1(x_k)} \frac{B_k}{\Delta_k} (M + \delta T) - \frac{2hA_k}{aJ_1(x_k)\Delta_k} T - \frac{B_k(q_k + p_k)}{\Delta_k} \tag{5}$$

Here

$$A_k = b_1 \frac{x_k^3}{a^3} + b_2\omega^2 \frac{x_k}{a} - b_3 \frac{x_k}{a}, \quad B_k = b_4 \frac{x_k^2}{a^2} + b_5\omega^2 + b_3 \tag{6}$$

$$C_k = c_1 \frac{x_k^4}{a^4} + c_2\omega^2 \frac{x_k^2}{a^2} + b_3 \frac{x_k^2}{a^2} - c_3\omega^2, \quad \Delta_k = B_k C_k - A_k^2$$

Expressions for φ_k are obtained from (5) by replacing B_k by A_k and A_k by C_k.

We determine the bending moment M and the tangential force T on the edge of the plate by satisfying the two remaining boundary conditions. To be specific, by considering the case when the edge is restrained $dw/dr = \varphi = 0$ when $r = a$, we get

$$M = \frac{a}{2h} \frac{\delta(S_2S_4 - S_1S_5 + \Sigma_j Q_{1j} p_j) + ha(S_2S_5 - S_3S_4 + \Sigma_j Q_{2j} p_j)}{S_2^2 - S_1S_3} \tag{7}$$

$$T = \frac{a}{2h} \frac{S_1S_5 - S_2S_4 + \Sigma_j Q_{3j} p_j}{S_2^2 - S_1S_3}$$

Here

$$S_1 = \Sigma_k \frac{x_k^2 B_k}{\Delta_k}, \quad S_2 = \Sigma_k \frac{x_k A_k}{\Delta_k}, \quad S_3 = \Sigma_k \frac{C_k}{\Delta_k} \tag{8}$$

$$S_4 = \Sigma_k \frac{B_k q_k x_k J_1(x_k)}{\Delta_k}, \quad S_5 = \Sigma_k \frac{A_k q_k J_1(x_k)}{\Delta_k}$$

$$Q_{1j} = S_2S_{4j} - S_1S_{5j},\ Q_{2j} = S_2S_{5j} - S_3S_{4j},\ Q_{3j} = S_1S_{5j} - S_2S_{4j}$$

The sums S_{4j} and S_{5j} are obtained from S_4 and S_5 on replacing q_k by z_{kj}.

By substituting expressions (7) into (5) and then into (2), we obtain an expression for the bending of the plate which can be written in the form

$$w = \Sigma_k (w_k' + \Sigma_j w_{kj}' p_j) J_0(x_k r/a) \tag{9}$$

The difference between the acoustic pressures acting at a point $r = (m-1)\Delta r$ on the two sides of the plate is expressed in terms of its deflection using a Huygens' integral /5/ and it is given by the formula /2/

$$p_m = \frac{\omega^2 \rho_0 n}{2\pi} \int_0^a \int_0^{2\pi} w(\rho) \frac{\exp(ik_0 R_m)}{R_m} \rho \, d\theta \, d\rho \tag{10}$$

$$R_m^2 = (m-1)^2 \Delta r^2 + \rho^2 - 2(m-1) \Delta r \rho \cos\theta$$

In the case of a plate-strip, the expression for the reaction of the medium at a point $x = (m-1)\Delta l$ has the form

$$p_m = \frac{i\omega^2 \rho_0 n}{2} \int_0^l w(x_1) H_0^{(1)}(k_0 (m-1) \Delta l - x_1)) \, dx_1 \tag{11}$$

In formulae (10) and (11), $k_0 = \omega/c_0$ is the wavenumber, ρ_0 and c_0 are the fluid parameters and $H_0^{(1)}$ is a Hankel function. The coefficient n is equal to 1 or 2 when there is a fluid on one or on two sides of the plate.

By putting $m = 1, 2, \ldots, N+1$ in formulae (10), we arrive at a system of algebraic equations for finding the nodal values of the acoustic pressure on the surface of the plate (δ_{mj} is the Kronecker delta)

$$\Sigma_j a_{mj} p_j = b_m; \; m = 1, 2, \ldots, N+1 \tag{12}$$

$$a_{mj} = \Sigma_k w'_{kj} I_{km} - \frac{2\pi}{\omega^2 \rho_0 n} \delta_{mj}, \quad b_m = -\Sigma_k w_k' I_{km} \tag{13}$$

$$I_{km} = \int_0^a \int_0^{2\pi} J_0\left(x_k \frac{r}{a}\right) \frac{\exp(ik_0 R_m)}{R_m} \rho \, d\theta \, d\rho$$

By separating the real and imaginary parts in Eqs.(12) and solving the resulting system of $2(N+1)$ equations, we find the real and imaginary components of the acoustic pressure at the nodes. We then calculate the deflection of the plate using formulae (9).

We will now present the results of calculations of the one-dimensional hydroelastic vibrations of three-layered plates which are excited by concentrated loads. The integrals I_{km} in (13) were represented as the sum of the integrals over the individual segments of the plate on each of which they were calculated using Gaussian quadrature formulae. The accuracy of the results obtained was checked by varying the number of segments into which the plate was subdivided and the number of terms which were retained in the series. As a numerical analysis shows, their values should be chosen using the formulae $N = 6n, k = n+2$, where n is the number of the closest resonant frequency. A further increase in the above-mentioned parameters had practically no effect on the results in the case of the examples being considered.

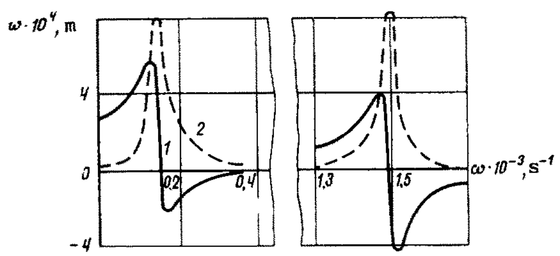

Fig.1

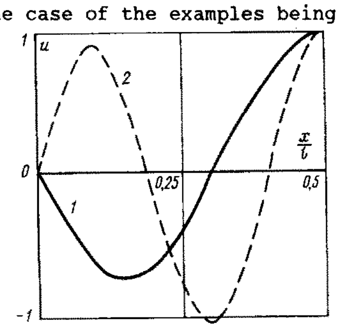

Fig.2

The frequency dependences of the real and imaginary components (curves *1* and *2*) of the deflection at the centre of a three-layered plate strip which was freely supported along its edges are shown in Fig.1. This plate strip had a width $l = 0.5$ m and consisted of aluminium facings with a thickness $\delta = 3 \times 10^{-3}$ m and a polyvinylchloride (PVC) filler ($E' = 5 \times 10^7$ Nm^{-2}, $\nu = 0.4$, $\rho = 0.3 \times 10^3$ kgm^{-3}) with a thickness $2h = 2 \times 10^{-2}$m. The plate was excited by a concentrated force $F = 10^3 \delta(x - l/2)$ Nm^{-1}. There was one-sided contact with water. By virtue of the symmetry of the load with respect to the centre of the plate, there were no even resonances and the fifth resonance occurring beyond the limits of the figure is observed at a frequency $\omega = 3702$ c^{-1}. Here, the modulus of the deflection attains the value $|w|_5 = 2.758 \times 10^{-4}$ m.

The difference between the third and the fifth modes of resonant vibrations and the natural modes of these vibrations *in vacuo* are illustrated by graphs 1 and 2 in Fig.2 which show the change in the imaginary part, which dominates in the resonance, the bending $u = \mathrm{Im}\, w(x)/|w(l/2)|$ on the surface of the plate. The corresponding resonant frequencies of the vibrations *in vacuo* (the characteristic frequencies of the plate) are: $\{\omega_1, \omega_3, \omega_5\} = \{854, 3065,$

6021} s^{-1}.

It is seen from the results which have been presented that, in the case of the hydroelastic vibrations of three-layered plates, the same features are retained as those noted in /2, 3/ in the case of homogeneous plates. At the same time, there are a number of differences. To begin with, the decrease in the amplitude of the resonance vibrations as the number of the resonance increases is significantly slower than in the case of homogeneous plates.

For comparison, we point to the fact that, in the case of a homogeneous aluminium plate of the same weight per unit length and width, the deflection at the centre at the resonant frequencies $\{\omega_1, \omega_3, \omega_5\} = \{84, 2230, 7670\}\ s^{-1}$ attained the values $\{|w|_1, |w|_3, |w_5|\} = \{26.32, 2.884, 0.4016\} \times 10^{-4}$ m. The above-mentioned effect showed up most clearly when the width (radius) of the plate was reduced, the thickness was increased and the rigidity of the filler was reduced, that is, when the parameters are varied in such a way as to be accompanied by an increase in the effect of shear deformations on the transverse vibrations.

Table 1

№	Value of parameter being varied	ω_1, s^{-1}	$\|w\|_1 \cdot 10^8$, m	ω_2, s^{-1}	$\|w\|_2 \cdot 10^8$, m	ω_3, s^{-1}	$\|w\|_3 \cdot 10^8$, m
1	—	(288) (798)	2226	1160 (2058)	1353	2312 (3420)	960
2	$a = 0.3$ m	894 (1878)	1096	3158 (4534)	1030	5874 (7418)	799
3	$\delta = 0.15 \cdot 10^{-2}$ m	240 (704)	4196	1038 (1942)	1818	2148 (3308)	1149
4	$2h = 1.5 \cdot 10^{-2}$ m	198 (672)	7145	892 (1870)	3063	1902 (3224)	1817
5	$E' = 1,25 \cdot 10^8$ Nm^{-2}	224 (622)	4393	870 (1536)	3426	1704 (3534)	2462
6	$\rho = 0.75 \cdot 10^3$ kgm^{-3}	294 (1004)	2052	1238 (2590)	1212	2540 (4300)	837
7	$E_1' = 0.36 \cdot 10^{11}$ Nm^{-2}	250 (690)	3672	1070 (1906)	1617	2198 (3254)	1031
8	$\rho_1 = 1.35 \cdot 10^3$ kgm^{-3}	290 (858)	2167	1186 (2210)	1303	2388 (3670)	917

This can be seen by analysing the amplitudes of the resonant hydroelastic vibrations of circular three-layered plates when they are excited by a concentrated force $F = 10$ N at the centre which have been presented in Table 1. Construction 1 consisted of aluminium layers with a thickness $\delta = 3 \times 10^{-2}$ m and an EK filler ($E' = 25 \times 10^7$ Nm^{-2}, $\nu = 0.4$, $\rho = 1.5 \times 10^3$ kgm^{-3}) with a thickness $2h = 3 \times 10^{-2}$ m and a radius $a = 0.6$ m. The plate was restrained along its periphery and there was a one-sided contact with water. The subsequent constructions differed in any one parameter. The value of the parameter which was varied (the index 1 refers to the facings) is given in the second column while the values of the first three resonant frequencies and deflections at the centre are given in the subsequent columns. The numbers in brackets are the resonant frequencies of the vibrations of the plate *in vacuo*.

It should be noted that the effect of a change in the plate parameters is most pronounced in the case of its first resonance. Its shift to lower frequencies as a consequence of a reduction in the thickness and stiffness of the facings or the filler is accompanied by an increase in the amplitude of the resonant vibrations. A reduction in the radius of the plate and the density of the filler and facing materials give rise to the reverse effect.

The effect of the damping properties of the filler on the resonant vibrations of three-layered plate-strip which has been considered above was investigated by introducing the complex elastic constants $\bar{E} = E(1 - i\eta)$ and $\bar{G} = G(1 - i\eta)$.

When the amplitudes of the resonant vibrations shown below for different values of the loss factor η

η	0	0.01	0.1	0.5
$\|w\|_1 \times 10^4$, m	8.06	7.87	6.44	3.59
$\|w\|_2 \times 10^4$, m	7.98	6.52	2.47	0.69
$\|w\|_3 \times 10^4$, m	2.76	1.8	0.662	0.181

are compared, it is seen that a substantial reduction in them is obtained at high values (10^{-2} - 10^{-1}) of η, when the damping of the vibrations due to the scattering of energy is comparable with their damping by the fluid.

In order to estimate the effect of the asymmetry of the structure of the plate on the resonant vibrations at a constant overall thickness of its facings, a solution of the problem being considered was obtained using the equations for the vibrations of an asymmetric three-layered plate strip presented in /4/. As a result of these calculations it was established that the first resonant frequency of plates with an asymmetric structure lie somewhat lower than in the case of plates with the same thickness of the facings and the amplitude of their vibrations is greater here, particularly in the case of a heavy filler. The values of the subsequent resonant frequencies were found to be smaller, while the amplitudes of the vibrations are greater in the case of plates with a symmetric structure and, here, this effect is most pronounced in the case of plates with a light, low stiffness filler.

In particular, when the thicknesses of the facings of the plate strip which has been considered above are changed to the vales $\delta_1 = 5 \times 10^{-3}$ m and $\delta_2 = 10^{-3}$ m, the following values of the resonant frequencies and deflections at the centre were obtained: $\{\omega_1, \omega_3, \omega_5\} = \{144, 1708, 4626\}$ s^{-1}, $\{|w|_1, |w|_3, |w|_5\} = \{8.705, 5.841, 1.516\} \times 10^{-4}$ m.

In concluding, we note that the investigations which have been carried out (some of the results have been presented above in the text) showed that the resonant frequencies of three-layered plates lie below the corresponding resonant frequencies of homogeneous plates with the same weight per unit length, while the amplitudes of the vibrations in them are several times greater. An exception is the first resonance of plates with light fillers of the polyvinylchloride (PVC) type, which is shifted to higher frequencies, while the amplitude of the vibrations is smaller.

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